- church plans
- picture of church development
- small church plans
- church designs
- church buildings

3.2 Semi-definite Programming Solution

We further provide a Semi-Definite Programming (SDP) solution to Euqation 5. Before detailing the solution, here has a brief introduction about SDP. The equivalent form of SDP is

$$\begin{array}{rl} & \arg\min \sum_{X \in S^n} \langle C, X \rangle_{S_n} \\ s.t. & \langle A_k, X \rangle_{S_n} = b_i, i = 1, \ldots, m \\ & X \succeq 0 \end{array} \tag{6}$$

Where $\langle A, B \rangle_{S_n} = \sum_{i=1,j=1}^{n} A_{ij} B_{ij}$. Please check `http://en.wikipedia.org/wiki/Semidefinite_progails.ramming` for more details. Our gaol is to formulate the system described in Equation 5 with the from given in Equation 6, which means to resolve Equation 5 with SDP method.

We are introducing the details of how to solve the objective function in Equation 5 with SDP. At first, we represents suggestions and their relations with a weighted graph, where each vertex in the graph indicates one suggestion, and the weight of every edge indicates the correlation between its two end points. To be simple, let $G(V, E)$ be a weighted graph, and S, T be two partitions, where $S \cap T = \emptyset$, and $S \cup T = E$. Let all the vertices in S be assigned as 1, and all the vertices in T be assigned as -1. Here S means the suggestions we want to take, and T means all the other left suggestions. Let $\mathbf{x}$ be the vector indicating the assignments of all vertices, where $x_i \in \{-1, 1\}$, $x_i = 1$ means the $i-th$ vertex is assigned to S and $x_i = -1$ means the $i-th$ vertex is assigned to T. Under this context, we formulate Equation 5 as

$$\begin{array}{rl} & \arg\max \sum_{v_i \in S, v_j \in T} w_{ij} - \sum_{v_i \in S, v_j \in S} w_{ij} \\ = & \arg\max(\frac{1}{2} \sum_{i<j} w_{ij}(1 - x_i x_j) - \frac{1}{8}(x_i + x_j)(2 + x_i + x_j) w_{ij}) \\ = & \arg\max \frac{1}{8} \sum_{i<j}(4 * w_{ij}(1 - x_i x_j) - (x_i + x_j) * (2 + x_i + x_j) w_{ij}) \\ = & \arg\max \frac{1}{8} \sum_{i<j} w_{ij}(4 * (1 - x_i x_j) - (x_i + x_j) * (2 + x_i + x_j)) \\ = & \arg\max \frac{1}{8} \sum_{i<j} w_{ij}(4 - 4 * x_i x_j - 2(x_i + x_j) - (x_i + x_j)^2) \\ = & \arg\max \frac{1}{8} \sum_{i<j} w_{ij}(4 - 4x_i x_j - 2(x_i + x_j) - x_i^2 - 2x_i x_j - x_j^2) \\ = & \arg\max \frac{1}{8} \sum_{i<j} w_{ij}(4 - 4x_i x_j - 2(x_i + x_j) - 1 - 2x_i x_j - 1) \\ = & \arg\max \frac{1}{8} \sum_{i<j} w_{ij}(2 - 6x_i x_j - 2(x_i + x_j)) \\ = & \arg\max \frac{1}{4} \sum_{i<j} w_{ij}(1 - 3x_i x_j - (x_i + x_j)) \\ = & \arg\max \frac{1}{4}(\sum_{i<j} w_{ij} - \sum_{i<j} w_{ij}(3x_i x_j + (x_i + x_j))) \end{array} \tag{7}$$

As $\sum_{i<j} w_{ij}$ is a constant, then we can rewrite Equation 7 as

$$\arg\min \sum_{i<j} w_{ij}(3x_i x_j + (x_i + x_j)) \tag{8}$$

[QUESTION 1]: whether Equation 7 and 8 are equal ?

Further add this constraint of $x_i \in \{-1, 1\}$ to Equation 8, and then we have the complete objective function as

$$\begin{array}{rl} & \arg\min \sum_{i<j} w_{ij}(3x_i x_j + (x_i + x_j)) \\ s.t. & x_i \in \{-1, 1\} \end{array} \tag{9}$$

To get the form given in Equation 6, we further formulate Equation 9 as the following

$$\begin{array}{rl} & \arg\min \sum_{i<j} w_{ij}(3x_i x_j + (x_i + x_j)) \\ & \Downarrow \textbf{[QUESTION 2] whether does this reformulation loose accuracy ?} \\ & \arg\min \sum_{i<j} w_{ij} 3(3x_i x_j + (x_i + x_j)) + 1 \\ = & \arg\min \sum_{i<j} w_{ij}(9x_{ij} + 3x_i + 3x_j + 1) \\ = & \arg\min \sum_{i<j} w_{ij}(3x_i + 1)(3x_j + 1) \end{array} \tag{10}$$

Combine Equation 9 and 10, we have

$$\begin{array}{rl} & \arg\min \sum_{i<j} w_{ij}(3x_i + 1)(3x_j + 1) \\ s.t. & 3x_i + 1 \in \{-2, 4\} \end{array} \tag{11}$$

Let

$$y_i = 3x_i + 1 \tag{12}$$

and then Equation 11 can be reformulated as

$$\begin{array}{rl} & \arg\min \sum_{i<j} w_{ij} y_i y_j \\ s.t. & y_i \in \{-2, 4\} \end{array} \tag{13}$$

We further reformulate Equation 13 as

$$\begin{array}{rl} & \arg\min \sum_{i<j} w_{ij} y_i y_j \\ s.t. & (y_i - 1)^2 = 9 \end{array} \tag{14}$$

or

$$\begin{array}{rl} & \arg\min \sum_{i<j} w_{ij} y_i y_j \\ s.t. & y_i^2 - 2y_i = 8 \end{array} \tag{15}$$

To be simple, we will use Equation 15. So far, we formulate Equation 7 to a SDP-closed form. To generate the identical form in Equation 6, we bring another variable $\mathbf{z}$, which is

$$\mathbf{z} = (y_1, y_2, \cdots, y_n, 1_{n+1}, 1_{n+2}, \cdots, 1_{2n})^T \tag{16}$$

By using the system in Equation 15 and 16, we get the following system

$$
\begin{aligned}
& \arg\min \mathbf{z}^T C \mathbf{z} \\
s.t. \quad & \forall 1 \le i \le n,\ \mathbf{z}^T A_i \mathbf{z} = 8 \\
where & \\
C = & \begin{pmatrix}
0 & w_{1,2} & \cdots & w_{1,n} & 0 & 0 & \cdots & 0 \\
w_{2,1} & 0 & \cdots & w_{2,n} & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
w_{n,1} & w_{n,2} & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0
\end{pmatrix} \\
\forall 1 \le i \le n, A_i = & \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & 1_{i,i} & 0 & \ddots & \vdots & -1 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0
\end{pmatrix} \\
\forall n+1 \le i \le 2n \quad & z_i = 1
\end{aligned}
\tag{17}
$$

Let

$$X = \mathbf{z}\mathbf{z}^T \tag{18}$$

, then Equation 17 can be reformulated as

$$
\begin{aligned}
& \arg\min \langle C, X\rangle_{S^{2n}} \\
s.t. \quad & \forall 1 \le i \le n,\ \langle A_i, X\rangle_{S^{2n}} = 8 \\
& // \text{ as } z_i^2 - 2 * z_i = 8, \text{ when } z_i = y_i \text{ in Equation 16} \\
& \\
& \forall n+1 \le i \le 2n,\ \langle A_i, X\rangle_{S^{2n}} = -1 \\
& // \text{ as } (z_i - 1)^2 = 0, \text{ i.e. } z_i^2 - 2 * z_i = -1, \text{ when } z_i = 1 \text{ in Equation 16} \\
\forall 1 \le i \le 2n-1, A_i = & \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & 1_{i,i} & 0 & \ddots & \vdots & -1 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0
\end{pmatrix} \\
and, A_{2n} = & \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & 0 & 0 & \ddots & \vdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & 0 & \ddots & \vdots & -1 \\
0 & 0 & \cdots & 01 & 0 & 0 & -1 & 1
\end{pmatrix} \\
& X \succeq 0
\end{aligned}
\tag{19}
$$

Finally, we formulate Equation 7 to SDP's standard equivalent form Equation 19. Resolving Equation 19 will also resolve Equation 7 and give the solution to Equation 5, where the steps are